Back to the subseismic approximation for core undertones

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The problem of the long-period gravity modes of the Earth outer fluid core (the core undertones) is investigated using either the subseismic or the anelastic approximation. These two approximations aim at filtering out the uninteresting short-period seismic or acoustic oscillations while taking into account the density variations across the core. However, they differ on the form of the equation of mass conservation since the density perturbations do not contribute to the mass balance in the anelastic case. Here we show that these two approximations lead to the almost same results because of the weakness of the core stratification due to the convective mixing. The anelastic approximation should be however preferred because it is simpler, mathematically self-consistent and is the only one which can be applied to problems with a stronger stratification such as the ones encountered in radiative zones of stars.

1 Introduction

To study the long-periods oscillations of the Earth outer core, geophysicists commonly use the so-called "subseismic approximation". This approximation has been proposed by Smylie and Rochester (1981) to reduce the order of the non-rotating governing equations from fourth to second. It aims at filtering out the high-frequency seismic P-waves from dynamics while taking into account the density variations across the core. Using the classical Boussinesq approximation for this problem is indeed quite satisfactory since, although the acoustic-like waves are filtered out, the equilibrium density is assumed to be constant (Spiegel and Veronis, 1960).

A similar approximation, the so-called "anelastic approximation", is used in meteorology and astrophysics to describe compressible convection. This approximation has first been derived by Batchelor (1953) and Ogura and Phillips (1962) to study the dry convection of the Earth troposphere. Their aim was to

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eliminate the sound waves from the hydrodynamic equations since these rapid waves impose very small time steps in the numerical integration. In addition, they wanted to take into account the density variations across the atmosphere so the Boussinesq approximation was still too restrictive. Quite naturally, this anelastic approximation has been used in astrophysics to the convective zones of stars (**?**, see e.g.)]Gil81. This approximation has also been used to describe the long-period oscillations propagating in the Earth atmosphere and star interiors. For example, Durran (1989) calculated the low-frequency oscillations of an isothermal atmosphere using the anelastic equations of Ogura and Phillips (1962). In the same way, Dintrans and Rieutord (2000) computed the oscillations of rapidly rotating stars. In order to decrease the size of the numerical rotating problem, a good solution indeed consists to filter out acoustic quantities while taking into account the large density variations encountered in radiative zones of stars.

The subseismic approximation has also been used in the field of stellar oscillations to describe the long-period gravity modes propagating in non-rotating stars. De Boeck et al. (1992) studied these modes by using the same subseismic set of equations as in Smylie and Rochester (1981). More recently, Dintrans and Rieutord (2001) proposed a comparative study of the anelastic and subseismic approximations applied to this problem (hereafter referred to as Paper I). There we showed that the subseismic approximation is not in fact a consistent approximation for the equations of motion. This result has been asserted by comparing the anelastic and subseismic eigenfrequencies with the complete ones in the case of the homogeneous star and the polytrope $n = 3$. Analytic solutions have been derived for the homogeneous case and the anelastic eigenfrequencies appear to be about twenty times more precise than the subseismic ones. In the same way, the computed anelastic eigenfrequencies are of about five times more precise than the subseismic ones in the polytropic case. We therefore concluded on the better efficiency of the anelastic approximation as far as stellar low-frequency oscillations are concerned.

The aim of this paper is to compare now both approximations in a geophysical context, namely that of the long-period oscillations of the Earth outer core. The subseismic approximation has been already tested in core dynamics and the resulting errors appear to be always at most $\mathcal{O}(1\%)$ (Smylie et al., 1984; Crossley et al., 1991; Crossley and Rochester, 1992). No comparative study with the anelastic approximation has been however made for this problem. In particular, the real benefit of the using of the subseismic approximation instead of the anelastic one is not clear. In fact, we will show that both approximations lead to the same results for the core undertones. It means that, contrary to stars, the inconsistency of the subseismic approximation does not induce unfortunate consequences in this case. Using the anelastic approximation is however preferable because it is self-consistent, simpler and can be successfully extended to more stratified astrophysical problems.

The plan of the paper is the following: after setting out the basic equations governing the adiabatic oscillations of a non-rotating fluid (Section 2.1), we recall the main features of the subseismic and anelastic approximations (Section 2.2) and give some details on the numerical method used to solve the oscillation equations (Section 2.3). The results of our calculations are then presented with the comparisons of the subseismic and anelastic eigenperiods with the complete ones (Section 3). We conclude with some outlooks of our results, with in particular their immediate implication to the more difficult rotating problem (Section 4).

2 The formalism

2.1 The equilibrium model and the complete oscillation equations

The identification by seismology of stable stratification zones in the Earth outer fluid core is still today controversial. Two stable stratified zones are in fact expected: *(i)* one zone may be located at its bottom (ICB) due to the release of light elements by the crystallization of the inner core (Souriau and Poupinet, 1991); *(ii)* a second zone may be located at its top (CMB) due to a thermal buoyancy flux coming from the mantle (Lister and Buffet, 1998). From observations of normal mode eigenperiods, Masters (1979) constrained the minimum value of the core Brunt-Väisälä period to be 6h, a value consistent with the core mixing due to convection (Crossley, 1984; Smylie et al., 1984). Thus many authors (Crossley and Rochester, 1980; Crossley et al., 1991; Crossley and Rochester, 1992) chose to take a uniform Brunt-Väisälä period of 6h in their studies of core dynamics and this model has also been used here.

Assuming a time-dependence of the form $\exp(i\sigma t)$ and neglecting rotation, the adiabatic oscillations of a self-gravitating fluid around its reference state obey the following linearized equations

$$\sigma^2\vec{\xi} = \vec{\nabla}\left(\frac{P'}{\rho_0} + \phi'\right) + N^2\left(\xi_r - \frac{P'}{\rho_0 g_0}\right)\vec{e}_r, \tag{1}$$

$$\frac{P'}{\rho_0} + c_0^2 \operatorname{div}\vec{\xi} - g_0\xi_r = 0, \tag{2}$$

$$\Delta\phi' = 4\pi G\rho_0\left(\frac{P'}{\rho_0 c_0^2} + \frac{N^2}{g_0}\xi_r\right). \tag{3}$$

where the variables P' and ϕ' are respectively the Eulerian perturbations of pressure and gravitational potential and $\vec{\xi}$ is the Lagrangian displacement.

The radial profiles ρ_0, g_0 and c_0^2 respectively denotes the density, gravity and square of seismic P-wave velocity and are given by the stable 6h core model of Crossley and Rochester (1992) with

$$N^2 = -g_0 \left(\frac{g_0}{c_0^2} + \frac{1}{\rho_0} \frac{d\rho_0}{dr} \right) = \left(\frac{2\pi}{6 \text{ hr}} \right)^2 = 8.461 \times 10^{-8} \text{ s}^{-2}. \tag{4}$$

As in Crossley and Rochester (1980), we assume rigid boundaries both at the ICB ($r = a$) and CMB ($r = b$) since eigenfrequencies computed using such conditions are in excellent agreement with those computed for the whole Earth (**?**, see also)]Cro91. In addition, the gravitational potential should be continuous across these surfaces which leads, for a mode of degree ℓ, to the following boundary conditions

$$\begin{cases} \xi_r = \dfrac{d\phi'}{dr} - \dfrac{\ell}{r}\phi' = 0 \quad \text{at} \quad r = a, \\[2ex] \xi_r = \dfrac{d\phi'}{dr} + \dfrac{\ell(\ell+1)}{r}\phi' = 0 \quad \text{at} \quad r = b. \end{cases} \tag{5}$$

2.2 The inconsistency of the subseismic approximation

The anelastic and subseismic approximations are based on the same assumption which is that the Eulerian fluctuations P' do not contribute to the Lagrangian ones δP, that is

$$\delta P = P' + \frac{dP_0}{dr}\xi_r = P' - \rho_0 g_0 \xi_r \simeq -\rho_0 g_0 \xi_r \quad \text{if} \quad \frac{P'}{\rho_0 g_0} \ll \xi_r. \tag{6}$$

Quite surprisingly, this common basic assumption does not lead to the same approximated form of the equation of mass conservation since

$$\text{Anelastic: } \operatorname{div} \vec{\xi} + \frac{d \ln \rho_0}{dr}\xi_r = 0, \quad \text{Subseismic: } \operatorname{div} \vec{\xi} - \frac{g_0}{c_0^2}\xi_r = 0. \tag{7}$$

We showed in I that the subseismic form of this equation is in fact incompatible with the basic assumption (6) since the neglect of P' in δP also implies the neglect of ρ' in $\delta\rho$, i.e.

$$P' \ll \frac{dP_0}{dr}\xi_r \Rightarrow \rho' \ll \frac{d\rho_0}{dr}\xi_r \quad \text{or} \quad \delta P \simeq \frac{dP_0}{dr}\xi_r \Rightarrow \delta\rho \simeq \frac{d\rho_0}{dr}\xi_r.$$

As a consequence, the only consistent form of the equation of mass conservation is the anelastic one according to

$$\rho' + \frac{d\rho_0}{dr}\xi_r + \rho_0 \operatorname{div}\vec{\xi} = 0 \Rightarrow \operatorname{div}\vec{\xi} + \frac{d\ln\rho_0}{dr}\xi_r = 0 \quad \text{or} \quad \operatorname{div}(\rho_0\vec{\xi}) = 0.$$

The density variations must be neglected in the mass balance if (6) is assumed.

An other important result derived in I concerns the uncoupling of the gravity perturbations ϕ'. Assuming (6), we indeed deduce the following system from Eqs. (1), (7) and boundary conditions (5)

$$\begin{cases} \sigma^2\vec{\nabla}\times\vec{\xi} = \vec{\nabla}\times(N^2\xi_r\vec{e}_r), \\ \\ \operatorname{div}\vec{\xi} + \dfrac{d\ln\rho_0}{dr}\xi_r = 0 \quad \text{or} \quad \operatorname{div}\vec{\xi} - \dfrac{g_0}{c_0^2}\xi_r = 0, \\ \\ \xi_r(a) = \xi_r(b) = 0. \end{cases} \tag{8}$$

This second-order set of equations valid for $\vec{\xi}$ and σ^2 clearly forms a well-posed eigenvalue problem so that the subseismic and anelastic eigenfrequencies can be computed without the taking into account of the ϕ' perturbations.

2.3 Numerics

As mentioned in Section 2.1, we choose the same equilibrium model as in Crossley and Rochester (1992), that is a stable core with a Brunt-Väisälä period of 6hr. This constant period gives us a natural time scale for our problem whereas the CMB radius b will be the length scale. Then the dimensionless perturbed fields are expanded onto spherical harmonics with, for example,

$$\vec{\xi}(r,\theta,\varphi) = \sum_{\ell=0}^{+\infty} u_\ell(r)Y_\ell\vec{e}_r + v_\ell(r)\vec{\nabla}Y_\ell, \quad \frac{P'}{\rho_0}(r,\theta,\varphi) = \sum_{\ell=0}^{+\infty} p_\ell(r)Y_\ell,$$

where $Y_\ell(\theta,\varphi)$ denotes the normalized spherical harmonic of degree ℓ with a zero azimuth m (**?**, $m = 0$ and no toroidal component needs to be taken into account since rotation is neglected; see e.g)]Rie91. As an example, the dimensionless form of the approximated set (8) reads

$$\begin{cases} \lambda^2 \left(u_\ell - v_\ell - x\dfrac{dv_\ell}{dx} \right) = u_\ell, \\ \\ x\dfrac{du_\ell}{dx} + (2+\beta x)u_\ell - \ell(\ell+1)v_\ell = 0, \qquad \left(\beta = \dfrac{d\ln\rho_0}{dx} \quad \text{or} \quad \beta = -\dfrac{g_0}{c_0^2} \right), \\ \\ u_\ell(\eta) = u_\ell(1) = 0, \end{cases}$$

where $\eta = a/b$ and $\lambda^2 = \sigma^2/N^2$. This system may be formally written as

$$\mathcal{M}_A \overrightarrow{\Psi}_{\ell n} = \lambda^2_{\ell n} \mathcal{M}_B \overrightarrow{\Psi}_{\ell n} \quad \text{with} \quad \overrightarrow{\Psi}_{\ell n} = \begin{pmatrix} u_{\ell n} \\ \\ v_{\ell n} \end{pmatrix},$$

where $\mathcal{M}_A$ and $\mathcal{M}_B$ denote two differential operators and $\overrightarrow{\Psi}_{\ell n}$ is the eigenvector associated with the eigenvalue $\lambda^2_{\ell n}$ of order n. This differential eigenvalue problem is discretized on the Gauss-Lobatto grid associated with Chebyshev's polynomials. Eigenvalues and eigenvectors are then computed with the iterative Arnoldi-Chebyshev solver already used to calculate the oscillations of rotating stars (**?**, for more details on numerics see)]Din99,Din00.

3 Results

In order to compare both approximations, we first calculated the eigenfrequencies of the complete problem consisting of Eqs. (1-3) plus boundary conditions (5). Then, we compared them with their anelastic and subseismic counterparts computed from (8). Table 1 shows the obtained periods (in hours) for some of the twenty first undertones of degree $\ell = 2$, the periods being deduced from the dimensionless eigenvalues $\lambda^2_{\ell n}$ using $T_{\ell n}(\text{hr}) = 6/\sqrt{\lambda^2_{\ell n}}$.

The first point is that the lower undertone period $T_1 \simeq 9.6705$ hr well agrees with that computed by Crossley and Rochester (1992) who found $T_1 \simeq 9.6741$ hr. The slight difference may be explained both by our different boundary conditions and the eigenvalue solver since these authors did not assume rigid boundaries and used a shooting method to compute eigenvalues.

The second point is that the anelastic and subseismic approximated periods are almost the same. This can be easily understood from the different forms

Table 1
Periods (in hours) of some of the twenty first $\ell = 2$ undertones of the stable 6hr core, with their corresponding anelastic and subseismic counterparts and the involved errors in percent.

n	Complete	Anelastic	Errors (%)	Subseismic	Errors (%)
1	9.6705371	9.6302857	0.4162	9.6328597	0.3896
3	23.031630	23.012466	0.0832	23.013470	0.0788
5	37.465939	37.453979	0.0319	37.454590	0.0303
10	74.139875	74.134154	0.0077	74.134462	0.0073
15	110.98888	110.98481	0.0036	110.98502	0.0035
20	147.88150	147.87863	0.0002	147.87878	0.0002

(7) of the equation of mass conservation where the ξ_r-term is either equal to $(d\ln\rho_0/dr)\xi_r$ or $-(g_0/c_0^2)\xi_r$. Because the Brunt-Väisälä frequency is weak, these two terms are almost equal; indeed from (4), we have

$$\frac{g_0}{c_0^2} + \frac{d\ln\rho_0}{dr} = -\frac{N^2}{g_0} \sim -\frac{8.5\times 10^{-8}\ \mathrm{s}^{-2}}{780\ \mathrm{cm/s}^2} \sim -10^{-5}\ \mathrm{km}^{-1}.$$

The difference in accuracy between both approximations thus depends on the stratification strength. As the Earth outer core is almost adiabatically stratified due to convection, the subseismic and anelastic approximations give the same results, which are very close to the exact ones as shown by Table 1. As expected, the relative errors decrease with increasing order of the mode. The accuracy level of both approximations is remarkable since, except for the first undertone, relative errors are less than 0.1% ! In fact, this very good agreement is still related to the weak stratification. When applying the anelastic approximation, one should indeed be careful about the required physical assumptions. Batchelor (1953) and Ogura and Phillips (1962) derived their original anelastic set of equations using two assumptions:

(i) the reference state around which oscillations occur is almost adiabatically stratified.
(ii) the time scale of any disturbance is similar to that of gravity waves.

Among these two basic assumptions, the second one is the easiest to fulfil. In this paper, we chose the constant Brunt-Väisälä period of our equilibrium model as the time scale. It automatically satisfies the second assumption since the computation is restricted to disturbances such as $\partial/\partial t \sim N$.

The first assumption is more severe. Assuming an almost adiabatic state means that the equilibrium potential temperature θ_0 is also nearly constant; i.e.

$\varepsilon = \delta\theta_0/\Theta \ll 1$ where $\delta\theta_0$ is the jump of the equilibrium potential temperature accross the layer and Θ is the reference mean value. Ogura and Phillips (1962) expanded all dependent variables as a power serie of ε and obtained the anelastic equations by collecting the first-order terms. The neglected terms are then $\mathcal{O}(\varepsilon^2)$ and give the accuracy of the development. This small dimensionless parameter ε may be related to the Brunt-Väisälä frequency by

$$N^2 = g_0 \frac{d \ln \theta_0}{dz} \sim \frac{g_0}{d}\frac{\delta\theta_0}{\Theta} \sim \frac{g_0}{d}\varepsilon,$$

where d denotes the layer thickness. In our case, we have

$$\varepsilon \sim \frac{d}{g_0} N^2 \sim \frac{2000 \text{ km}}{780 \text{ cm/s}^2} 8.5 \times 10^{-8} \text{ s}^{-2} \sim 2\%.$$

Hence the mean error made when using the anelastic approximation on the core undertones is about $\mathcal{O}(\varepsilon^2) \sim 0.04\%$ which agrees well with the results from Table 1. The long-period Earth core oscillations can therefore be seen as a good case for the anelastic approximation and, despite it inconsistency, also the subseismic approximation since the almost neutral stratification prevents unfortunate consequences in this case. On the contrary, as the stratification increases like for instance in stellar radiative zones, these two approximations disagree more and more and the anelastic results become more accurate as shown in I.

4 Conclusion

We have computed the complete, anelastic and subseismic approximated eigenperiods of a stable 6h Earth's outer core. After setting out the complete equations governing the adiabatic oscillations of a non-rotating and self-gravitating fluid, we presented the basic properties of the anelastic and subseismic approximations.

Both approximations assume that the Eulerian pressure perturbations P' do not contribute to the Lagrangian ones δP, that is only the fluctuations stemming from the equilibrium pressure gradient are retained. We then recalled that, unlike the anelastic case where the perturbed density field has no influence on the mass balance, the subseismic equation of mass conservation is not compatible with this assumption (this result being demonstrated in I). As a consequence, the subseismic approximation is not a consistent approximation of the equations of motion.

We also recalled that the gravity perturbations decouple from the motion in both cases. It refutes the common belief in the respect of self-gravity by the subseismic set of equations; i.e. the only difference with the anelastic approximation lies in the equation of mass conservation. Then, two main results have been enlightened by our computations:

– for the Earth's fluid outer core, anelastic and subseismic eigenperiods are very close to each other. We explained this result by the very weak stratification of the fluid. The differences between the anelastic and subseismic periods indeed come from the stratification strength. Because of convection, the Earth core is almost adiabatically stratified and both approximations give the same results.
– the anelastic relative errors are very small ($\lesssim$0.1%) for core undertones. It is not obvious at first sight since the anelastic set of equations has been initially derived for the atmospheric dry convection. We explained this very good agreement by recalling the required physical assumptions implied by the use of the anelastic approximation: *(i)* the reference state should be almost adiabatically stratified; *(ii)* the time scale of any disturbance should be similar to that of gravity waves. The core undertones clearly fulfil very well these two basic assumptions and the anelastic eigenperiods are thus accurate. Because of the weakness of the Brunt-Väisälä frequency, the subseismic results are the same as the anelastic ones and both approximations can be in fact applied to *this problem*. However, it is more natural to employ the anelastic approximation since it is simpler, mathematically self-consistent and it is the only one which can be successfully applied to problems with a stronger stratification such as the long-period oscillations of stars.

The above results have straightforward consequences as far as geophysical applications are concerned. The main application concerns the rotating problem, that is to calculate the eigenperiods of the rotating outer fluid core. Dintrans et al. (1999) already calculated the gravito-inertial waves propagating in a rapidly rotating stratified shell which had the same aspect ratio than the Earth outer core (i.e. $\eta = a/b \simeq 0.35$). To decrease the size of the numerical problem, this work has been done under the Boussinesq approximation so that the density variations across the outer core have been neglected. Hence an improvement of this problem is now possible by taking the more complete equilibrium model used in this paper by the means of the anelastic approximation.

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